Fixpoint Theorem

Fixpoint theorem

- One form of *Knaster-Tarski Theorem* (1955):
- If D is a domain and $f : D \to D$ is monotonic, then f has at least one fixpoint
- More interesting consequence:
- If $\bot$ is the least element of D, then f has a *least fixpoint*, and that fixpoint is the largest element in the chain
- $\bot, f(\bot), f(f(\bot)), f(f(f(\bot))) \ldots f^n(\bot)$
- Least fixpoint: a fixpoint of f, x such that, if y is a fixpoint of f, then $x \sqsubseteq y$

Examples

- For domain of powersets, { } is the least element
- For identity function, $f^n(\{\ \})$ is the chain
- { }, { }, { }, ... so least fixpoint is { }, which is correct
- For $f(x) = x \cup \{a\}$, we get the chain
- { }, {a}, {a}, ... so least fixpoint is {a}, which is correct
- For $f(x) = \{a\} - x$, function is not monotonic, so not guaranteed to have a fixpoint!
- Important observation: as soon as the chain repeats, we have found the fixpoint (why?)

Proof of fixpoint theorem

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  - $\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq \ldots \sqsubseteq f^k(\bot) = f^{k+1}(\bot) = \ldots$

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- Second, prove that $f^n(\bot)$ is the *least* fixpoint
  - Let p be a fixpoint: $f(p) = p$

    $\bot \sqsubseteq p$

    $f(\bot) \sqsubseteq f(p)$

    …

    $f^k(\bot) \sqsubseteq f^k(p) = p$

Adding a top

- Now let us consider domains with an element $\top$, such that for every point x in the domain, $x \sqsubseteq \top$
- New theorem: if *D* is a domain with a greatest element $\top$ and $f : D \to D$ is monotonic, then the equation x = f(x) has a *greatest* solution, and that solution is the smallest element in the sequence
- $\top$, $f(\top)$, $f(f(\top))$, ...
- Proof?

Multi-argument functions

- If D is a domain, a function $f : D \times D \rightarrow D$ is monotonic if it is monotonic in each argument when the other is held constant
- Intuition:
  - Electrical circuit has two inputs
  - If you raise either input while holding the other constant, the output either goes up or stays the same

Fixpoints of multi-arg functions

- Can generalize fixpoint theorem in a straightforward way
- If D is a domain and $f, g : D \times D \to D$ are monotonic, the following system of equations has a least fixpoint solution, calculated in the obvious way
- $x = f(x, y)$ and $y = g(x, y)$
- Can generalize this to more than two variables and domains with greatest elements easily

next: lattices