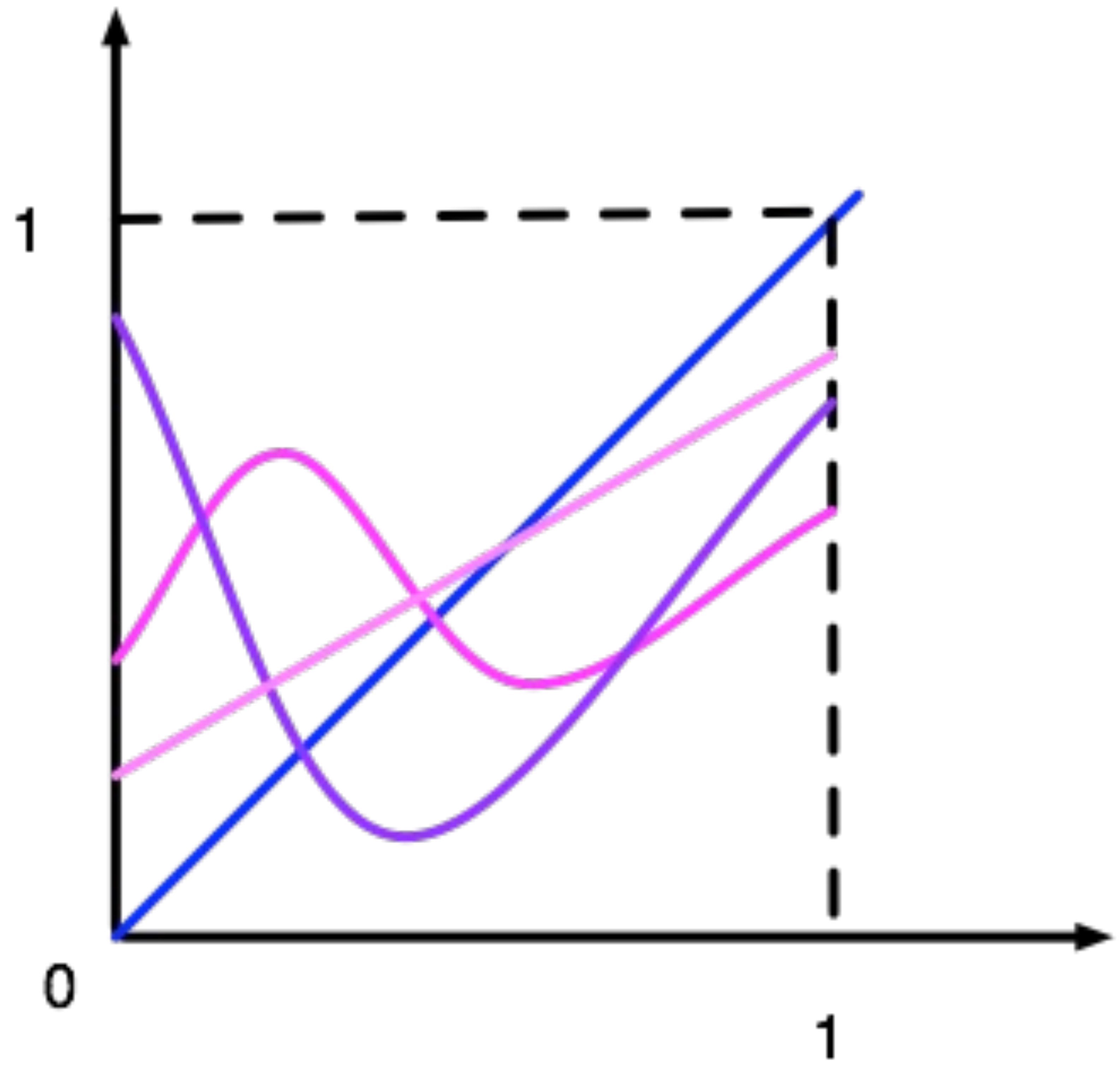
Lattice Theory

First, something interesting

- Brouwer Fixed point Theorem
  - Every continuous function *f* from a closed disk into itself has at least one fixed point
- More formally:
  - Domain D: a *convex, closed, bounded* subspace in a plane (generalizes to higher dimensions)
  - Function $f : D \to D$
  - There exists some x such that $f(x) = x$

Intuition

- Consider the one-dimensional case: mapping a line segment onto itself
- $x \in [0, 1]$
- $f(x) \in [0, 1]$
- There must exist some x for which $f(x) = x$
- Examples (in 2D)
  - A mall directory
  - Crumpling up a piece of graph paper

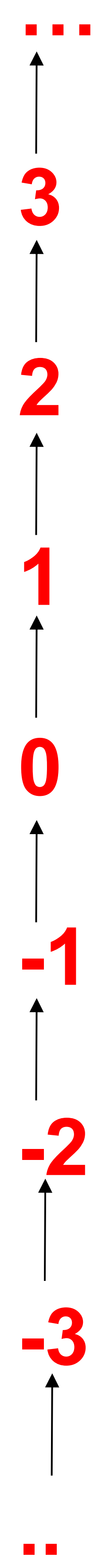


Game plan

- Finite partially ordered set: D
- Function $f : D \to D$
- Monotonic function $f : D \to D$
- $\exists$ fixpoint of f
  - $\exists$ *least* fixpoint of f
- Generalization to case when D has a greatest element, $\top$
  - $\exists$ *greatest* fixpoint of f
- Generalization to systems of equations

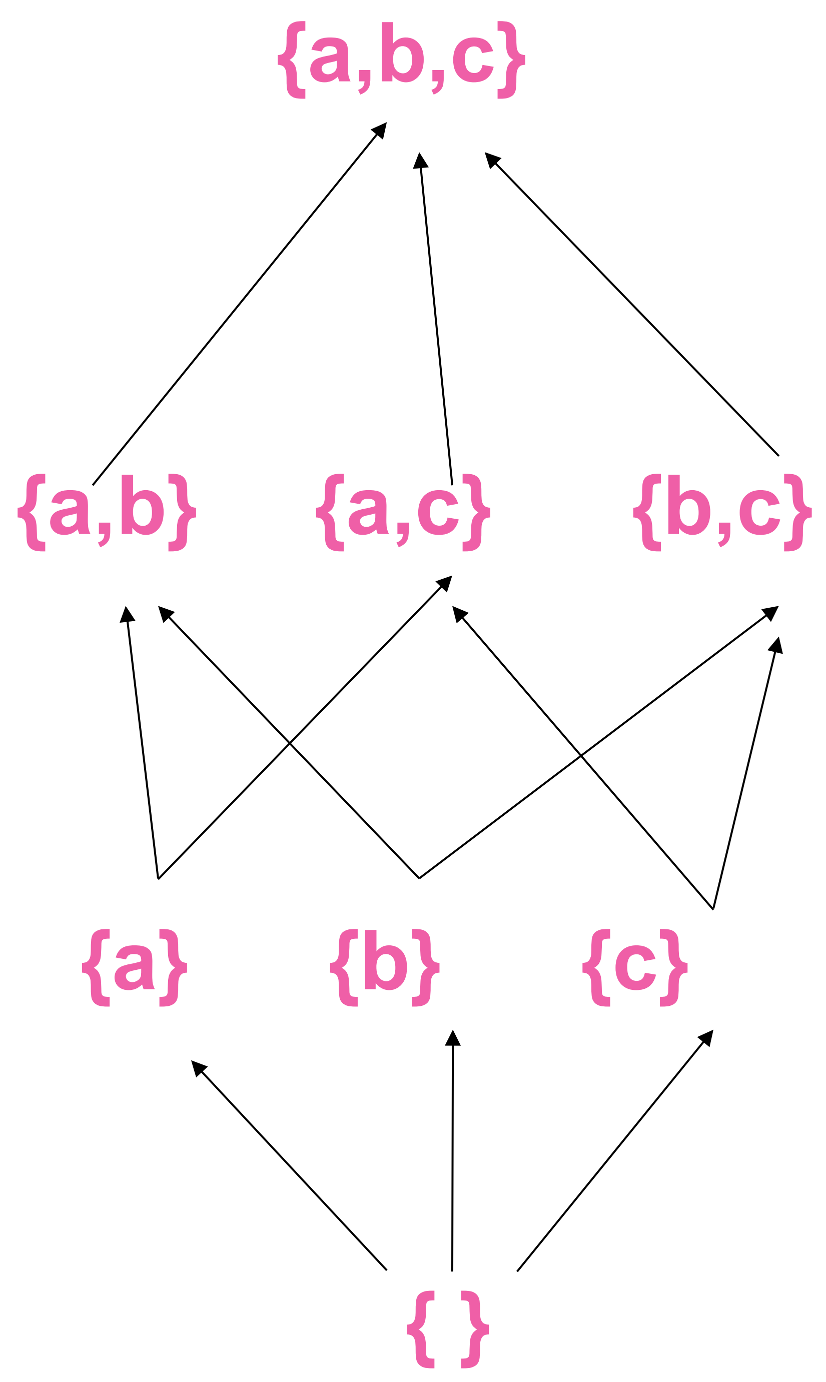
Partially ordered set (poset)

- Set *D* with a relation $\sqsubseteq$ that is
  - Reflexive: $x \sqsubseteq x$
  - Anti-symmetric: $x \sqsubseteq y$ and $y \sqsubseteq x \Rightarrow y = x$
  - Transitive: $x \sqsubseteq y$, $y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- Example: set of integers and **≤**
- Graphical representation of poset
  - Graph in which nodes are elements of *D* and relation $\sqsubseteq$ is indicated by arrows
  - Usually omit reflexive and transitive arrows for legibility
  - Not counting reflexive edges, graph is always a DAG (why?)



Another example

- Powerset of any set, ordered by ⊆ is a poset
- In the example, poset elements are {}, {a}, {a, b}, {a, b, c}, etc.
- X ⊑ Y iff X ⊆ Y



Finite poset with least element

- Poset in which
  - Set is **finite**
  - There is a **least** element that is below all other elements in poset
- Examples
  - Set of integers ordered by $\leq$ is *not* a finite poset with least element (no least element, not finite)
  - Set of natural numbers ordered by $\leq$ has a least element (0), but not finite
  - Set of factors of 12, ordered by $\leq$ has a least element as is finite
  - Powerset example from before is finite (how many elements?) with a least element ({})

Domains

- "Finite poset with least element" is a mouthful, so we will abbreviate this to *domain*
- Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis
  - (Goal: what is a lattice?)

next: functions over domains