Lattice Theory
First, something interesting

• Brouwer Fixed point Theorem

• Every continuous function $f$ from a closed disk into itself has at least one fixed point

• More formally:
  
  • Domain $D$: a convex, closed, bounded subspace in a plane (generalizes to higher dimensions)
  
  • Function $f : D \rightarrow D$

• There exists some $x$ such that $f(x) = x$
• Consider the one-dimensional case: mapping a line segment onto itself

• \( x \in [0, 1] \)

• \( f(x) \in [0, 1] \)

• There must exist some \( x \) for which \( f(x) = x \)

• Examples (in 2D)
  • A mall directory
  • Crumpling up a piece of graph paper
Game plan

• Finite partially ordered set: \( D \)
• Function \( f : D \rightarrow D \)
• Monotonic function \( f : D \rightarrow D \)
• \( \exists \) fixpoint of \( f \)
  • \( \exists \) least fixpoint of \( f \)
• Generalization to case when \( D \) has a greatest element, \( T \)
  • \( \exists \) greatest fixpoint of \( f \)
• Generalization to systems of equations
Partially ordered set (poset)

- Set $D$ with a relation $\sqsubseteq$ that is
  - Reflexive: $x \sqsubseteq x$
  - Anti-symmetric: $x \sqsubseteq y$ and $y \sqsubseteq x \Rightarrow y = x$
  - Transitive: $x \sqsubseteq y$, $y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- Example: set of integers and $\leq$
- Graphical representation of poset
  - Graph in which nodes are elements of $D$ and relation $\sqsubseteq$ is indicated by arrows
  - Usually omit reflexive and transitive arrows for legibility
  - Not counting reflexive edges, graph is always a DAG (why?)
Another example

- Powerset of any set, ordered by $\subseteq$ is a poset
- In the example, poset elements are $\emptyset$, \{a\}, \{a, b\}, \{a, b, c\}, etc.
- $X \subseteq Y$ iff $X \subseteq Y$
Finite poset with least element

• Poset in which
  • Set is **finite**
  • There is a **least** element that is below all other elements in poset
• Examples
  • Set of integers ordered by $\leq$ is *not* a finite poset with least element (no least element, not finite)
  • Set of natural numbers ordered by $\leq$ has a least element (0), but not finite
  • Set of factors of 12, ordered by $\leq$ has a least element as is finite
  • Powerset example from before is finite (how many elements?) with a least element (\{}\)
Domains

• “Finite poset with least element” is a mouthful, so we will abbreviate this to *domain*

• Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis

• (Goal: what is a lattice?)
next: functions over domains